Holomorphic families of knots

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Abstract
A classical example of failure of Newlander–Nirenberg theorem in infinite dimensions, due to LeBrun, is the space of unparametrized knots in a conformally Riemannian 3-dimensional manifold. We study the possible compact analytic sumbanifolds of finite dimension in such manifolds.

Contents

1 Introduction

1.1 Complex structure on the space of knots

Let S^1 be a circle, M a smooth manifold, and $\mathrm{Maps}(S^1, M) = \{f \mid f\colon S^1 \to M\}$ the set of smooth maps equipped with Fréchet topology of convergence with all derivatives. It is a manifold which can be modelled locally on the vector space $\Gamma(S^1, f^*TM)$ of vector fields defined along the image of f. This Fréchet manifolds carries a natural action of the group $D = \mathrm{Diffeo}^+(S^1)$ of orientation-preserving diffeomorphisms of circle by $\delta(f) = f \circ \delta^{-1}$. The quotient is again a manifold, with tangent space at $[f]$ isomorphic to $\Gamma(S^1, f^*TM)/\Gamma(S^1, TS^1)$, which is isomorphic to $\Gamma(S^1, \nu_f)$, where the normal bundle ν_f is defined as f^*TM/TS^1, due to the Serre–Swan theorem.

Definition 1.1: The quotient $\mathrm{Maps}(S^1, M)/D$ is called the **space of knots** in M and defined $\mathrm{Kn}(M)$.

Provided M is endowed with a conformally Riemannian structure q, so are endowed the normal spaces to knots. In particular, when $\dim M = 3$, a conformal structure on the normal bundle is the same as a complex structure operator, hence q gives rise to the complex structure operator on any space $\Gamma(S^1, \nu_f)$ and thus an almost complex structure operator on $\operatorname{Kn}(M)$.

Theorem 1.2: (LeBrun [LeBrun], Lempert [Lempert]). This almost complex structure has vanishing Nijenhuis tensor. However, it is not induced by any complex analytic atlas.

Definition 1.3: Let $p \in M$ be a point. The locus of knots passing through p is a complex submanifold of $\operatorname{Kn}(M)$ of codimension one, called the **polar divisor** and denoted by Π_p.

Proof: If $f(z) = p$ for $z \in S^1$, then the condition on the tangent subspace to this locus is $\{v \in \Gamma(S^1, \nu_f) \mid v(z) = 0\}$, which is a complex-linear condition. ■

Remark 1.4: The space of knots is not merely a complex manifold: it carries a natural real structure (that is, an anti-holomorphic involution) defined by $\overline{f}(z) = f(z^{-1}), z \in S^1$, which reverts the orientation of a knot. It is easy to see that this involution has no fixed points and preserves the polar divisors.

1.2 CR manifolds and LeBrun's CR twistors

Definition 1.5: Let N be a smooth manifold of real dimension $2n+1$. An **almost CR structure** is a rank n subbundle $H^{1,0} \subset TN \otimes \mathbb{C}$ with property $H^{1,0} \cap \overline{H^{1,0}} = 0$. It is called **integrable**, or simply **CR structure**, if one has $\left[H^{1,0}, H^{1,0}\right] \subset H^{1,0}$. The real distribution $H \subset TN$ spanned by $H^{1,0}$ and $H^{0,1} = \overline{H^{1,0}}$ is called **horizontal**, it carries a complex structure operator defined by $\sqrt{-1}$ on $H^{1,0}$ and $-\sqrt{-1}$ on $H^{0,1}$. If $(N, H \subset TN)$ is a contact manifold, then a CR structure is said to be **supported** on H, if H is its horizontal distribution. A horizontal map $(N, H) \xrightarrow{f} (N,' H')$ (that is, such that $(df)(H) \subset H'$) is called **(anti-)CR holomorphic** iff its derivative induces a complex (anti)linear map of horizontal bundles.

Definition 1.6: Let (N, H) be a contact manifold, and $H^{1,0} \subset H \otimes \mathbb{C}$ is an integrable CR structure supported on it. Since that $[H^{1,0}, H^{1,0}] \subseteq H^{1,0}$ and similarly for $H^{0,1}$, the Frobenius tensor $\Lambda^2 H \to TN/H$ reduces to a (1,1)-form $H^{1,0} \otimes H^{0,1} \to TN/H$ with coefficients in the quotient by the horizontal bundle. It is called the **Levi form**.

Example 1.7: Let (X, I) be an almost complex manifold, and $N \subset X$ a real codimension one submanifold. Let $H^{1,0} = T^{1,0} \cap TN$. Then it is an almost CR structure, supported on a corank one subbundle $H = TN \cap I(TN) \subset TN$. If the almost complex structure I is integrable, so is $H^{1,0}$. If $N \subset X$ is given as a zero locus $\{u = 0\}$, then $H = \ker(d^c u)|_N$, and the Levi form as $(dd^c u)|_N$.

Twistor corrsepondence between real four-dimensional manifolds and complex threefolds with certain geometric data, due to Penrose and Atiyah, is now classical. LeBrun employed CR manifolds to extend this correspondence to real three-dimensional manifolds. Indeed, if $M \subset M'$ is a hypersurface in a four-dimensional manifold, the restriction of the twistor fibration $\mathrm{Tw}(M')|_M$ is a real hypersurface in a complex threefold, carrying a natural CR structure. LeBrun showed that this construction can be furnished without any reference to an embedding.

Theorem 1.8: Let M be a compact three-dimensional manifold, ST^*M be the spherization of its cotangent bundle (that is, $T^*M \setminus 0_{T^*M}$ modulo rescaling), and D its natural horizontal distribution. Then there exists a natural construction which associates to any class of conformally Riemannian metric $[g]$ on M an operator $I_{[g]} \colon D \to D$ which makes ST^*M into a CR manifold, for which the fibration $ST^*M \to M$ is a foliation into holomoprhic curves. This CR manifold is denoted by $\mathrm{Tw}(M, [g])$ and called the **LeBrun's CR twistor space**.

In what follows, we shall denote the horizontal distribution on the LeBrun's twistor space by the Cyrillic letter D (traditionally called "dobro"), which was the LeBrun's choice in the very first paper on the subject, and reserve the overused letter H for some other occasions.

LeBrun's original construction was based on the bundle of null cones $Q = \{v \in T^*N \otimes \mathbb{C} \mid g(v,v) = 0\}$ inside the complexified tangent bundle of M, restriction of complex-valued symptectic form onto it and taking the Hamiltonian reduction. This immediately implies involutivity of the bundle $D^{1,0}$ and conformal invariance of this structure. Verbitsky proposed an alternative, more transparent construction, in which unfortunately the desired properties follow by nontrivial calculations.

Proposition 1.9: (Verbitsky [Verbitsky]) Let (M, g) be a three-dimensional Riemannian manifold, and $D \subset T\,(ST^*M)$ be the standard contact distribution. Let $D = D^v \oplus D^h$ be the splitting of it into vertical and horizontal part, induced by the Levi-Civita connection associated to g, in particular, $D^h_{p,\tau} \cong \tau \subset T_pM$. Let I_g be the complex structure on D, defined on D^v as the standard complex structure on the unit sphere, and on D^h as the rotation by $\pi/2$ in the positive direction in the corresponding oriented plane $\tau \subset T_pM$. Then I_g is the LeBrun's CR structure $I_{[g]}$.

Remark 1.10: It is clear though from this description that the twistor space $\mathrm{Tw}(M)$ carries a natural fixed-point-free anti-CR involution: namely, flipping the orientation of the plane $(p, \tau) \mapsto (p, \overline{\tau})$.

In the main part of the present paper, we shall introduce a third definition of the LeBrun's CR structure in spirit of Eells–Salamon. Now let us notice that the LeBrun's CR space is indeed a twistor space, that is, its CR geometry encodes the conformally Riemannian geometry of the three-dimensional manifold it emanates from.

Theorem 1.11: (LeBrun, [LeBrun, §4]) Let (N, D) be a 5-dimensional CR manifold with nondegenerate Levi form and a smooth foliation by $\mathbb{C}\mathrm{P}^1$s. Then the space M of its leaves is a real three-dimensional manifold. This manifold M admits a conformally Riemannian metric such that (N, D) is its LeBrun's twistor space $\mathrm{Tw}(M)$, iff N carries a CR contact form orthogonal to the foliation, and an anti-CR involution preserving the foliation and respecting the contact form.

[what follows should be a separate section in the middle of the paper, dedicated to our approach to the LeBrun's CR twistor space via umbilics; the present section should be a brief exposition of the existing knowledge on LeBrun's twistors]

1.3 Universal knot and its double fibration

Let us fix a point $o \in S^1$ once and for all, and let $D^\circ \subset D$ be the group of diffeomorphisms fixing o. Of course one has $D/D^\circ \cong S^1$.

Definition 1.12: The **space of marked knots**, or the **universal knot** $\mathrm{Kn}^\circ(X)$ is defined as the quotient $\mathrm{Maps}(S^1, M)/D^\circ$. It carries a structure of a double fibration: the fibration $\mathrm{fgt}\colon \mathrm{Kn}^\circ(X) \to \mathrm{Kn}(X)$ given by taking quotient further by D, thus with fiber $D/D^\circ \cong S^1$, which forgets the marked point, and another $\mathrm{ev}\colon \mathrm{Kn}^\circ(X) \to X$ given by $[f] \mapsto f(o) \in X$, called the **evaluation map**.

This double fibration allows a transgression operation $\mathrm{T} = \mathrm{fgt}_* \circ \mathrm{ev}^*$, which associates an object on the space of knots $\mathrm{Kn}(M)$ to an object on M. For example, the transgression of a point $p \in M$ is the polar divisor $\Pi_p \subset \mathrm{Kn}(M)$. If $\alpha \in \Omega^k(M)$ is a differential k-form, then $\mathrm{T}\alpha$ is a differential $(k-1)$-form on $\mathrm{Kn}(M)$ obtained by pulling α back to the universal knot and then integrating it over fibers. In the simplest case of a 1-form α its transgression $\mathrm{T}\alpha$ is a function given by $(\mathrm{T}\alpha)(f) = \int_{S^1} f^*\alpha$. Another important case is when α is a volume form on M. Then $\mathrm{T}\alpha \in \Omega^2(\mathrm{Kn}(M))$ is a nowhere degenerate 2-form. If $d\alpha = 0$, then $d\mathrm{T}\alpha = 0$ as well, since the transgression operation is defined on the topological level, and since top degree form is always closed, this means that $\mathrm{T}\alpha$ for a volume form α is a symplectic form on $\mathrm{Kn}(M)$.

Proposition 1.13: Let (M, g) be a Riemannian manifold, and β its Riemannian volume form. Then $\mathrm{T}\beta \in \Omega^2(\mathrm{Kn}(M))$ is a Kähler form for the complex structure on $\mathrm{Kn}(M)$ induced by the conformal class of $[g]$.

Proof: ■

Definition 1.14: Let $\Sigma \subset M$ be a surface. Consider the locus $\widetilde{\Pi}_\Sigma = \{f \in \mathrm{Kn}^\circ(M) \colon f(o) \in \Sigma, f'(o) \perp \Sigma\} \subset \mathrm{Kn}^\circ(M)$. Its projection $\Pi_\Sigma = \mathrm{fgt}\left(\widetilde{\Pi}_\Sigma\right) \subset \mathrm{Kn}(M)$ is called the **transpolar locus**.

Proposition 1.15: The projection $\widetilde{\Pi}_\Sigma \to \Pi_\Sigma$ is the normalization of simple normal crossing singularities. If $\Sigma \subset M$ is umbilic, then Π_Σ is a complex codimension one subvariety, called the **transpolar divisor**. Moreover, the evaluation map $\Pi_\Sigma \to \Sigma$ (strictly speaking defined only on the normalization $\widetilde{\Pi}_\Sigma$) is holomorphic w. r. t. the conformal structure on Σ restricted from M.

Remark 1.16: The polar divisors Π_p for $p \in M$ a point can be considered as degenerate cases of transpolar divisors, when the point p is viewed as a sphere of vanishing radius—hence the name. However, a transpolar divisor is not generally speaking preserved by the real structure on $\mathrm{Kn}(M)$.

Proof: The first assertion is obvious: the preimage of a knot from Π_Σ is a pair of a knot $\gamma \in \Pi_\Sigma$ and a point from $\gamma \cap \Sigma$ where the intersection is orthogonal. There is a finite number of such points on a knot, and two such points cannot collide, since otherwise a tangent vector to Σ formed by colliding points would be perpendicular to Σ at the same time. Generically, there is only one such point.

The second assertion amounts to an annoying coördinate computation involving local curvatures, which is way below the authors' skills and dignity. ■

1.4 CR structure on the universal knot

The evaluation map $\mathrm{Kn}^\circ(M) \to M$ gives rise to a contact distribution. Namely, if $f \in \mathrm{Kn}^\circ(M)$ is a marked knot, then one can define the horizontal subspace $H_f \subset T_f \mathrm{Kn}^\circ$ as the inverse image $H_f = \mathrm{ev}^* \nu_{f(o)}$ of the normal subspace $\nu_{f(o)} f(S^1) \subset T_{f(o)} M$. This distribution has real codimension one and is transversal to the fibers of the projection $\mathrm{Kn}^\circ(M) \to \mathrm{Kn}(M)$, hence projects onto the tangent spaces of $\mathrm{Kn}(M)$ isomorphically, and inherit the formally integrable complex structure.

The evaluation map can be refined using the LeBrun's twistor space.

Definition 1.17: Let M be a conformally Riemannian 3-dimensional manifold, and $\mathrm{Tw}(M) = ST^*M$ be its LeBrun's CR twistor space. The map $\mathrm{vel} \colon \mathrm{Kn}^\circ(X) \to \mathrm{Tw}(M)$ which sends a marked knot f to the oriented plane $\nu_{f(o)} f(S^1) \subset T_{f(o)} M$, is called the **velocity map**.

The rationale for the name is as follows: in presence of a conformal structure, the bundles ST^*M and STM are identified, and the normal plane to a knot at its marked point corresponds to the positive direction tangent to the knot at the marked point.

Definition 1.18: The pullback $H \subset T\,\mathrm{Kn}^\circ(M)$ of the contact distribution on $\mathrm{Tw}(M)$ under the velocity map $\mathrm{vel}\colon \mathrm{Kn}^\circ(M) \to \mathrm{Tw}(M)$ is called the **natural contact distribution**. Since the projection $d\,\mathrm{fgt}\colon H \to T\,\mathrm{Kn}(M)$ is a fiberwise isomorphism, the pullback of the almost integrable complex structure $I\colon T\,\mathrm{Kn}(M) \to T\,\mathrm{Kn}(M)$ defines a natural CR structure on $\mathrm{Kn}^\circ(M)$ supported on H, which we shall refer to as the **natural CR structure**. The real structure on $\mathrm{Kn}(M)$ extends obviously to an anti-CR involution on $\mathrm{Kn}^\circ(M)$.

Proposition 1.19: The velocity map $\mathrm{vel}\colon \mathrm{Kn}^\circ(M) \to \mathrm{Tw}(M)$ is horizontal, CR holomorphic, and intertwines the natural anti-CR involutions.

Proof: The horizontality assertion follows from the definition of the contact distribution H. To prove CR holomorphicity, pick up a point $p \in M$, $\tau \subset T_pM$ an oriented plane, and $\ell \subset H_{p,\tau}\,\mathrm{Tw}(M)$ a complex line (w. r. t. the LeBrun's CR structure on $\mathrm{Tw}(M)$). By the Umbilic Point lemma (Lemma 2.10), there exists a surface $\Sigma \subset M$ passing through p and umbilic at p with $T_p\Sigma = \tau$, such that $T_{p,\tau}(ß\Sigma) = \ell \subset T_{p,\tau}\,\mathrm{Tw}(M)$, where $ß\colon \Sigma \to \mathrm{Tw}(M)$ is the Gauss map. We shrink Σ (to a formal germ of surface if needed) so that it is totally umbilic. The transpolar divisor Π_Σ is thus a (germ of a) complex analytic submanifold in $\mathrm{Kn}(M)$, and thus its lift $\Pi'_\Sigma \subset \mathrm{Kn}^\circ(M)$ is a horizontal CR holomorphic submanifold. By Proposition 1.15, the forgetful projection along this submanifold is holomrphic w. r. t. the induced conformal structure on Σ, and thus the restriction of the velocity map to it (it has range $ß\Sigma$) is also holomorphic, since holomorphic are the Gaussian lifts of totally umbilic surfaces (Proposition 2.8).

Thus the velocity map has complex linear differential whenever restricted to the preimage of any complex line in $H \subset T\,\mathrm{Tw}(M)$, which means that the velocity map is indeed CR holomorphic. The last assertion is immediate. ■

In particular, preimages of horizontal CR holomoprhic curves in $\mathrm{Tw}(M)$ under the velocity map project to complex analytic loci in $\mathrm{Kn}(M)$. For twistorial lines $ST^*_pM = \mathrm{Tw}_p(M) \subset \mathrm{Tw}(M)$, this yields polar divisors Π_p, and for the Gaussian lifts of totally umbilic surfaces—the transpolar divisors. Another example are preimages of mere points, i. e. fibers of the velocity map:

Definition 1.20: The fibers of the velocity map $\mathrm{Kn}^\circ \to \mathrm{Tw}(M)$, projected to $\mathrm{Kn}(M)$, are codimension two complex submanifolds called the **penicillar loci** and denoted by $\tau_{v,p}$. They parametrize the knots passing through p with tangent vector v.

2 LeBrun's CR structure via umbilics

Let us remind of the following classical construction from analytic mechanics.

Remark 2.1: Let X be a manifold, and $ST^*X \xrightarrow{\pi} X$ the spherization of its tangent bundle (quotient of T^*M with zero section removed by the rescaling action of $\mathbb{R}_{>0}$). Then it carries a natural contact distribution $D \subset T(ST^*X)$, defined by $D_{(\xi,p)} = \ker(\pi^*\xi)$, $\xi \in ST^*_pX$ a 1-form at p. The Frobenius tensor $\Lambda^2 D \to T/D$ defined by $u \wedge v \mapsto [u, v] \mod D$ defines a (conformally) symplectic structure on each horizontal space D, which we shall denote by λ.

Definition 2.2: Let $\Psi \subset X$ be a coöriented hypersurface. Then the map $ß_\Psi \colon \Psi \to ST^*X$, defined by $ß(p) = T_p\Psi \in ST^*_pX$, is called the **Gauss map**.

Proposition 2.3: The images of the Gauss map are horizontal and Lagrangian w.r.t. the Frobenius form.

Proof: The horizontality is immediate. Since $ß(\Psi)$ is a submanifold, the vector fields on it commute into vector fields on the same submanifold, and since it is horizontal, they contribute nothing to the Frobenius tensor. ■

Remark 2.4: If $\dim X = n+1$, then for a hypersurface with prescribed tangent hyperplane there are $n(n+1)/2$ independent quadratic terms of the Taylor expansion, and since all such hypersurfaces lift by the Gauss map into horizontal Lagrangian submanifolds with different tangent spaces, any horizontal Lagrangian subspace is a tangent of the Gauss image of some hypersurface (the dimension of the Lagrangian Grassmannian in $\mathbb{R}^{2n}$ equals exactly $n(n+1)/2$).

The horizontal space $D_{\xi,p}$ fits into the exact triple

$$0 \to T_\xi ST^*_pM \to D_{\xi,p} \to \ker\xi \subset T_pM \to 0,$$

and the subspace $T_\xi ST^*_pM \subset D_{\xi,p}$ is Lagrangian w.r.t. the Frobenius form λ. Note that if M carries a conformal structure, the left and right term both carry complex structure operator: one on the left since it is a tangent plane to a round sphere in a Euclidean 3-space, and one on the right because it is an oriented plane with a conformal structure.

Proposition 2.5: Let g be a Riemannian metric in our conformal class on M. There exists a unique complex structure operator on $D \subset T\,\mathrm{Tw}(M)$ with the following property: for any point $p \in M$ and any surface $\Sigma \subset M$ passing throug p such that $p \in \Sigma$ is an umbilic point, the tangent space $T_{p,T_p\Sigma}ß(\Sigma)$ to the Gauss lift $ß(\Sigma)$ is a complex line. This almost complex structure does not depend on a choice of a metric in a fixed conformal class.

Proof: Since the umbilicity is a second-order condition, and a second-order formal neighborhoods of any points in a Riemannian manifold are isometrically isomorphic thanks to the existence of geodesic normal coödrinates, one can think of a surface in a flat space, say given by the equation $z = \frac{ax^2+2bxy+cy^2}{2}$. The tangent plane to the point (x, y) of such surface is given by 1-form $(ax+by)dx+(bx+cy)dy-dz$, and hence the Gauss map can be written as

$$ß(x,y) = \left(x, y, \frac{ax^2+2bxy+cy^2}{2}, \frac{ax+by}{\sqrt{N}}, \frac{bx+cy}{\sqrt{N}}, \frac{-1}{\sqrt{N}}\right) \in T^*(\mathbb{R}^3),$$

where N stands for $(ax + by)^2 + (bx + cy)^2 + 1$. This looks ugly, yet we only need the first-order terms of the Taylor expansion at $(0, 0)$, so it boils down to

$$ß(\delta x, 0) = (\delta x, 0, 0, a\delta x, b\delta x, -1),$$

$$ß(0, \delta y) = (0, \delta y, 0, b\delta y, c\delta y, -1),$$

where $\delta^2 = 0$. Umbilicity of the surface at $(0, 0)$ implies that the eigenvalues of the matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ coincide, that is, $\det A = (\mathrm{tr}\ A)^2$, and elementary algebra implies that $a = c$ and $b = 0$. Thus the tangent vectors to the Gauss lift are $(\delta x, 0, 0, a\delta x, 0, -1)$ and $(0, \delta y, 0, 0, a\delta y, -1)$, and such subspaces for any a are indeed complex lines for a unique complex structure, which coincides with that described by Verbitsky (Proposition 1.9).

To conclude the proof, one can refer to a classical theorem of Schouten and Struik stating that the umbilic points are unchanged by the conformal modification of the metric on the ambient space. ■

Remark 2.6: One might put this as follows: a choice of a Riemannian metric determines the Levi-Civita connection, thus the Levi-Civita-horizontal distribution within D transversal to the fibers of the twistor projection $ST^*M \to M$, and in particular its $(1, 0)$-part. The Proposition 2.5 means that though this Levi-Civita-horizontal $(1, 0)$ distribution depends on the choice of the metric, its span with the vertical $(1, 0)$-distribution only depends on its conformal class.

Proposition 2.7: The distribution $D^{1,0}$ constructed above is involutive.

Proof: First, $[D^{1,0}, D^{1,0}] \subseteq D \otimes \mathbb{C}$ by construction: the Frobenius form has type $(1, 1)$, thus the commutator of any two $(1, 0)$ fields vanish after dividing by D. ??? ■

Proposition 2.8: Let M be a three-dimensional conformally Riemannian manifold, and I be some CR structure on the contact manifold ST^*M subject to following conditions:

1. The fibers of the projection $ST^*M \to M$ are complex curves,
2. For any surface $\Sigma \subset M$, taken with the induced conformal structure, the Gauss map $ß\colon s \mapsto (s, T_s\Sigma) \in ST^*M$ is holomorphic at s iff $s \in \Sigma$ is an umbilic point.

Then I is the LeBrun's CR structure.

Proof: Such CR structure, whenever exists, is unique: its $H^{1,0}$-subspase is spanned by the lifts of second-order germs of umbilic surfaces via the Gauss map at each point. Thus it suffies to show that the LeBrun's CR structure satisfies this condition.

The differential $d\text{ß}$ maps $T_s\Sigma$ to $T_{s,T_s\Sigma}ST^*M$, being a section of the differential of the projection $H_{s,T_s\Sigma} \to \Sigma$. The projection map is complex-linear, thus the image of the complex-antilinear part $\overline{\partial}\text{ß}$ lies in the vertical subbundle. A direct calculation shows that it equals to the Willmore integrand, which is known to admit an expression $(\varkappa_1 - \varkappa_2)^2 d\Sigma$ after trivializing the normal bundle, where $\varkappa_i$ are the principal curvatures of $\Sigma \subset M$. ■

Proposition 2.9: Let M be a three-dimensional conformally Riemannian manifold, $\mathrm{Tw}(M)$ its LeBrun's twistor space, and $\lambda \in \Lambda^{1,1}H^*$ its Levi form. For any surface $\Sigma \subset M$, the pullback $\text{ß}^*\lambda$ of the Levi form along the Gauss map is the Willmore integrand on Σ.

Proof: The Levi form measures the failure of $(1,0)$ and $(0,1)$ vector fields from $H \otimes \mathbb{C}$ to commute. As soon as Σ is considered with its conformal structure restricted from M, such fields commute there, so all the failure comes from the non-holomorphicity of the Gauss map, and thus, as we have seen it the proof of Proposition 2.8, agrees with the Willmore integrand. ■

Let us also state a partial converse to the Proposition 2.8, which would be important in what follows.

Lemma 2.10: Let M be a 3-dimensional conformally Riemannian manifold, $p \in M$ a point, $\tau \subset T_pM$ an oriented plane, and $\ell \subset H_{p,\tau} \subset T_{p,\tau}\,\mathrm{Tw}(M)$ a complex linear subspace which is not tangent to the twistorial line $ST^*_pM \subset \mathrm{Tw}(M)$. Then there exists a surface $\Sigma \subset M$ passing through p with $T_p\Sigma = \tau$, and with $T_{p,\tau}(\text{ß}\Sigma) = \ell$ (by Proposition 2.8, it implies that Σ is necessarily umbilic at p).

Proof: Take a Riemannian metric on M in the conformal class. Up to second order, M is formally Euclidean at p (in geodesic normal coördinates). In the Euclidean space, one can pass a sphere of any radius through a given point with a given tangent plane at it. This gives the first two terms in the Taylor expansion of Σ at p, and these are the only terms important to ensure umbilicity. ■

Remark 2.11: The complex linear susbpace tangent to $ST^*_pM \subset \mathrm{Tw}(M)$ can also be viewed as a tangent, namely to the sphere of vanishing radius, which we though do not normally consider as a legitimate "surface", let alone "umbilic".

Example 2.12: The LeBrun's CR twistor space of a conformally round 3-sphere arises naturally as a real hypersurface in the complex projective space $\mathbb{C}\mathrm{P}^3$. Namely, consider the quaternionic Hopf fibration

$$\mathbb{C}\mathrm{P}^3 = \mathrm{P}(\mathbb{C}^4) \cong \mathrm{P}_{\mathbb{C}}(\mathbb{H}^2) \to \mathbb{H}\mathrm{P}^1 \cong S^4.$$

If one embeds S^3 as an equator of S^4, then its preimage in $\mathbb{C}\mathrm{P}^3$ is a hypersurface fibered over M into straight lines, and its induced CR structure is CR isomorpic to the LeBrun's twistor space $\mathrm{Tw}(S^3)$.

More concretely, the Hopf fibration can be written as

$$(x : y : u : v) \mapsto (x + jy : u + jv)$$

for a choice of unit quaternion j anti-commuting with i. $S^3 \subset \mathbb{H}\mathrm{P}^1$ is given by the equation $\mathrm{Re}\left(\frac{x+jy}{u+jv}\right) = 0$, or, equivalently, $x'u'+x''u''+y'v'+y''v'' = 0$, where $x = x' + ix''$ etc. That is, $\mathrm{Tw}(S^3)$ is a real quardic hypersurface in the complex projective space $\mathbb{C}\mathrm{P}^3$. This equation is also equivalent to $\mathrm{Re}(x\overline{u} + y\overline{v}) = 0$; notice that the expression in brackets is not algebraic: if it were equivalent to $\mathrm{Re}\, f = 0$ for f a polynomial, the algebraic variety $\{f = 0\}$ would be an integral submanifold for the CR distribution on $\mathrm{Tw}(S^3)$, which is proscribed by the nondegeneracy of the Levi form.

Taking derivatives, one can conclude that the contact element at point $(x : y : u : v)$ is given by the equation $\overline{u}\delta x+\overline{v}\delta y+\overline{x}\delta u+\overline{y}\delta v = 0$, and since the real part of this condition defines the tangent bundle to the real quadric $\{\mathrm{Re}(x\overline{u} + y\overline{v}) = 0\}$, the Reeb vector field can be given by $\delta x = iu, \delta y = iv, \delta u = ix, \delta v = iy$.

3 Compact complex sumbanifolds in the space of knots

We shall start from a simple observation summing up the discussion in the introduction.

Proposition 3.1: Let M be a compact conformally Riemannian 3-dimensional manifold, and let $X \subset \mathrm{Kn}(M)$ be a compact complex submanifold in its space of knots (in particular, of finite dimension). Then X is a complex projective manifold.

Proof: The theorem of LeBrun (Theorem 1.2), together with the finite-dimensional Newlander–Nirenberg theorem, implies that the complex structure on X is integrable. Proposition 1.13 implies that X is Kähler. Moreover, the transgression operation preserves the integral structure on cohomology because of its topological nature; thus the transgression of the fundamental class of M restricts to an integral class in $H^2(X)$, and existence of an integral Kähler class implies projectivity by the Kodaira embedding theorem. ■

However,

Proposition 3.2: (maximum principle for holomorphic families of knots) Let $C \subset \mathrm{Kn}(M)$ be a compact complex curve, and $C^\circ \to C$ be the restriction of the universal knot to it. Then the evaluation map $\mathrm{ev}\,|_C \colon C^\circ \to M$ is surjective.

Proof: Let us examine what rank can the differential $d\mathrm{ev}|_C$ have. Note that it is always at least one: for any point $f \in C$, the restriction $\mathrm{ev}|_{\mathrm{fgt}^{-1}(f)}$ is an immersion since $\mathrm{fgt}^{-1}(f) = f(S^1)$ is always an immersed knot. In the direction of the horizontal distribution, $d\mathrm{ev}$ respects the complex structure, hence it can be either 0 or 2. In total, the possible rank is either 1 or 3, and equals 3 in generic point (otherwise the family of knots would be constant). The subset $Y \subset C^\circ$ where the rank of $d\mathrm{ev}|_C$ does not drop is an open dense subset.

Both C° and M are compact three-dimensional manifolds. The image $\mathrm{ev}(C^\circ) \subset M$ is closed, and it contains the image $\mathrm{ev}(Y)$, which is open since the evaluation map $\mathrm{ev}\,|_Y$ is étale, as a dense subset. If $M \setminus \mathrm{ev}(C^\circ)$ is nonempty, then it is open, and the boundary $\partial\,(\mathrm{ev}(C^\circ))$ is two-dimensional. But it can be only one-dimensional since otherwise the rank of $d\mathrm{ev}|_C$ equals 2 somewhere. ■

Proposition 3.3: Let M be a three-dimensional Riemannian manifold with Riemannian volume form β such that $\int_M \beta = 1$, $C \subset \mathrm{Kn}(M)$ a compact complex curve, and $C^\circ \to C$ the universal knot over it. Then the degree of C with respect to the Kähler form $\mathrm{T}\beta$ equals the degree of the evaluation map $\mathrm{ev}\,|_C \colon C^\circ \to M$. Moreover, if $\mathrm{vel}\colon C^\circ \to \mathrm{Tw}(M)$ is the velocity map, then this degree equals the intersection number of $\mathrm{vel}(C^\circ)$ with the fiber of the projection $\mathrm{Tw}(M) \xrightarrow{\pi} M$.

Proof: By definition of the transgression, one has $\int_C \mathrm{T}\beta = \int_{C^\circ} \mathrm{ev}^*\beta$. The latter integral equals $\deg(\mathrm{ev}|_C) \int_M \beta = \deg(\mathrm{ev}|_C)$. The evaluation map factorizes as $\mathrm{vel} \circ \pi$, where π is the twistor projection, and the second part of the Proposition then follows from the fact that $\pi^*[\beta] \in H^3(\mathrm{Tw}(M))$ is the Poincaré dual of the class of the fiber. ■

The Proposition 3.2 can be reformulated as follows: if $C \subset \mathrm{Kn}(M)$ is a compact complex curve, then through any point $p \in M$ passes a knot from C. It of course applies to surfaces, since they are all projective; moveover, it can be refined further. To state it, we need two or three more definitions.

Definition 3.4: Let M be a conformally Riemannian three-dimensional manifold. The **symmetric square of the universal knot**, defined by $\mathrm{Kn}^{\circ 2}(M)$, is the fiberwise symmetric square of the usual universal knot $\mathrm{Kn}^\circ(M) \to \mathrm{Kn}(M)$, a fibration into Möbius bands over $\mathrm{Kn}(M)$ with fiber parametrizing the pairs of (maybe colliding) points on the corresponding knot.

Definition 3.5: Let M be a conformally Riemannian three-dimensional manifold. Its **Hilbert square** $\mathrm{Hilb}^2(M)$ is the manifold with boundary, obtained from its symmetric square $\mathrm{Sym}^2(M)$ by blowing into diagonal the sphere bundle parametrizing the tangent directions. The **bi-evaluation map**

$$\mathrm{ev}^2\colon \mathrm{Kn}^{\circ 2}(M) \to \mathrm{Hilb}^2(M)$$

is defined by sending a pair of points to a pair of points, if the points are distinct, and by sending a pair of points on a knot into a positive tangent direction to this knot, if the points collided.

Proposition 3.6: Let $S \subset \mathrm{Kn}(M)$ be a compact complex surface. Then the restriction of the bi-evaluation map $\mathrm{ev}^2\colon S^{\circ 2} \to \mathrm{Hilb}^2(M)$ is surjective.

Proof: Let $p \in M$ be a point. By Proposition 3.2, any curve $C \subset S$ contains a knot passing through p. Thus varying C, we see that the intersection of S with the polar divisor Π_p is a curve $C_p \in S$. Now applying the Proposition 3.2 to the curve C_p, we see that any other point (including the ones infinitesimally close to p—that is, tangent directions in T_pM) can be joint with p by a knot from C_p (and thus from S). ■

We could proceed further by induction to prove that if $X \subset \mathrm{Kn}(M)$ is a compact complex n-fold, than through any n-tuple of points in M one can pass a knot from X. We shall not do this though, since this statement turns out to be vacuous:

Proposition 3.7: Let $X \subset \mathrm{Kn}(M)$ be a compact complex n-fold. Then $n \leqslant 2$.

The idea behind the proof is as follows. Similarly to what we did above, we could consider the n-th symmetric power of the universal knot, and n-th evaluation map from it into "$\mathrm{Hilb}^n(M)$," a manifold with corners that parametrizes n-tuples of points with collisions properly desingularized, and show that it must be a surjection. Thinking of n-tuples with possible collisions algebro-geometrically as of length n subschemes, one notices that "subschemes" coming from knots are "curvilinear" (something analogous to subschemes cut out from an algebraic curve), whereas "$\mathrm{Hilb}^n(M)$" for $n \geqslant 3$ contains "subschemes" which are not "curvilinear". However, making this argument rigourous would involve developing a scheme-theoretic view on conformally Riemannian threefolds, which is far beyond the scope of the present paper. Thus we present an Ersatz argument encapsulating the same sentiment.

Proof: By a similar inductive argument, one shows that for any pair of points $p, q \in M$ the locus of knots from X passing through both p and q is at least a curve. Taking a limit $q \to p$ along some direction $v \in T_pM$, one gets a compact complex curve $C \subset \tau_{v,p} \subset \mathrm{Kn}(M)$.

Pick up a metric of total volume 1 on M within the prescribed conformal class, and let β be its volume form. By Proposition 3.3, one has $\int_C \mathrm{T}\beta =$

$[\mathrm{vel}(C^\circ)] \cap [\pi^{-1}(p)] \in H_5(\mathrm{Tw}(M))$. In our situation, the circle bundle $C^\circ \to C$ admits a natural section $\sigma_0 \colon f \mapsto (f, p)$. In presence of metric, one can define a family of sections σ_ε obtained from σ_0 by positive rotation by ε (note that in general this is not a U(1)-action since the knots from C can have nonconstant lengths). Thus $\int_C \mathrm{T}\beta$ can be further rewritten as the linking number of $\pi^{-1}(p)$ and $\mathrm{vel}(\sigma_\varepsilon)$, two surfaces inside a 5-dimensional manifold $\mathrm{Tw}(M)$. However, if we make a local cut of M by a surface passing through p orthogonally to v, then for ε small enough the surfaces $\mathrm{ev}(\sigma_\varepsilon)$ would lie on one side of this local cut. Thus the linking number of $\pi^{-1}(p)$ and $\mathrm{vel}(\sigma_\varepsilon)$ must be zero. This is impossible, since it equals the integral of a Kähler form over a compact curve. ■

Remark 3.8: The above proof no longer works if one does not assume the curve to be compact. For example, take a 3-sphere S^3 with conformally round metric, and consider all circles (not just the great ones) passing through a point p and tangent to a tangent vector $v \in T_pS^3$. If one realizes S^3 as a Euclidean space compactified by a point at infinity (let it be p), then these circles become a pencil of parallel lines, which clearly form a complex curve in $\mathrm{Kn}(S^3)$ parametrized by $\mathbb{C}$.

Motivated by the above statement and the previous Remark, we make the following

Conjecture 3.9: Let M be a compact Riemannian three-dimensional manifold with Riemannian volume form β, $p \in M$ a point, $v \in T_pM$ a tangent vector, and $\tau_v \subset \mathrm{Kn}(M)$ the locus of knots passing through p with tangent vector v. Then the Kähler form $(\mathrm{T}\beta)|_{\tau_v}$ admits a Kähler potential: $(\mathrm{T}\beta)_{\tau_v} = dd^c u$ for some plurisubharmonic function $u \in C^\infty(\tau_v)$.

4 Case of the round sphere

To illustrate the generalities, we shall consider in greater detail the case already touched in Example 2.12 and Remark 3.8, the case of families of knots in a round sphere.

Proposition 4.1: Let S^3 be equipped with a round Riemannain metric, and $Q \subset \mathrm{Kn}(S^3)$ be the family of oriented great circles. Then Q is a complex submanifold biholomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$.

Proof: Let us realize the round S^3 as the unit sphere in the Euclidean space $\mathbb{R}^4$. Then an oriented great circle γ can be identified with the oriented 2-dimensional plane $\tau_\gamma \subset \mathbb{R}^4$ which cuts it out, so Q may be identified with the Grassmannian of oriented planes $\mathrm{Gr}(2,4)$. A normal vector field along circle γ pointing to a new circle γ', considered as a vector field within $\mathbb{R}^4$, is orthogonal to both γ and S^3, that is, orthogonal to τ_γ, and it is easy to see that it is a restriction of a linear vector field defined along τ_γ and pointing to $\tau_{\gamma'}$. Under this identification, the complex structure on $T_\gamma Q$ is identified with the complex structure on $T_\tau \mathrm{Gr}(2,4) = \mathrm{Hom}(\tau, \tau^\perp) = \tau^* \otimes \tau^\perp$ which arises from the natural identification $\tau^\perp \cong \mathbb{C}$.

The Grassmannian of oriented planes may be now identified with a smooth quardic surface as follows: for each plane τ, its complexification $\tau \otimes \mathbb{C}$ contains exactly two isotropic lines ℓ and $\overline{\ell}$, and choice of either corresponds to the choice of orientation on τ. This defines a map into the quadric $\{\ell \subset \mathbb{C}^4 \colon (\ell, \ell) = 0\} \subset \mathrm{P}(\mathbb{C}^4)$. It is again immediately holomorphic: the tangent space to the quadric is isomorphic to $\mathrm{Hom}(\ell, \ell^\perp/\ell)$, ■

Proposition 4.2: Under the identification of $\mathrm{Tw}(S^3)$ with both the unit tangent bundle UTS^3 and the universal great circle Q°, the geodesic flow corresponds to the U(1)-action on $Q^\circ \to Q$ by the rotation in positive direction.

Proof: ■

The late Proposition arouses temptation to conjecture that this must be the case for any compact complex surface in the space of knots. However, such an abortive conjecture might be disproved by essentially the same example:

Example 4.3: Let S^3 be equipped with the round metric, $Q \subset \mathrm{Kn}(S^3)$ the family of oriented great circles, and $\xi \colon S^3 \to S^3$ a conformal transformation which is not an isometry. Then $\xi Q \subset \mathrm{Kn}(S^3)$ is a compact complex surface on which the lenght functional associated to the Riemannian metric on S^3 is nonconstant.

Nevertheless, a following variation of this conjecture the authors were unable neither to prove nor to disprove:

Conjecture 4.4: Let M be a conformally Riemannian 3-dimensional manifold, and $S \subset \mathrm{Kn}(M)$ a compact complex surface in its space of knots. Then there exists a Riemannian metric in the chosen conformal class, in which all the knots from S have the same length.

5 Surfaces with rational polar curves

From the previous section, we learn that possible compact complex submanifolds in a space of knots are severely restricted: its maximal possible dimension is two. Now we examine the geometry of such possible surfaces, which we shall call, for the sake of brevity, the **surfaces of knots**.

Proposition 5.1: Let $S \subset \mathrm{Kn}(M)$ be a surface of knots, and $S^\circ \to S$ be the universal knot over it. Then the velocity map $\mathrm{vel} \colon S^\circ \to \mathrm{Tw}(M)$ is surjective.

Proof: Follows directly from Proposition 3.6. ■

Remark 5.2: This statement is in a way more important that the general Proposition it stems from, since the velocity map is horizontal and CR holomorphic (Proposition 1.19). Although both CR manifolds $\mathrm{Kn}^\circ(M)$ and $\mathrm{Tw}(M)$ seem to arise as boundaries of certain manifolds $\mathrm{Kn}^{\circ 2}(M)$ and $\mathrm{Hilb}^2(M)$, and the horizontal CR holomorphic map between them is induced by the bi-evaluation map $\mathrm{ev}^2\colon \mathrm{Kn}^{\circ 2}(M) \to \mathrm{Hilb}^2(M)$, we are not aware of any complex structures on either of these manifolds which would give rise to the CR structures on their boundaries being considered, and have no way to assert any holomorphicity of the bi-evaluation map.

Thus for a surface of knots $S \subset \mathrm{Kn}(M)$, the velocity map $\mathrm{vel}\colon S^\circ \to \mathrm{Tw}(M)$ is a surjective horizontal CR holomorphic map between two CR manifolds of the same dimension, nondegenerate in the direction transversal to the horizontal distribution. It may not be a ramified cover though, since possibility of a blowdown cannot be ruled out.

Proposition 5.3: Let S be a surface of knots. Then the Chern class $c_1(S^\circ \to S)$ of the universal knot (considered as a fibration into circles) in nonzero.

Proof: Why indeed not? ■

Remark 5.4: An example of a surface S with vanishing Chern class $c_1(S^\circ \to S)$ would be a surface of knots contained entirely in a polar divisor. Thus the Proposition 5.3 may be viewed as a refinement of the argument from the proof of Proposition 3.7.

Definition 5.5: Let $S \subset \mathrm{Kn}(M)$ be a surface of knots. Preimages of twistorial lines $ST^*_p(M) \subset \mathrm{Tw}(M)$, projected to S, are called the **polar curves** and denoted by C_p. It is of course the same as curves cut out by polar divisors $\Pi_p \subset \mathrm{Kn}(M)$ on S.

The polar curves play a rôle similar to poles and polars in projective duality, but now between a three-dimensional manifold M and a surface of knots S: a point $p \in M$ gives rise to a curve $C_p \in M$, and a point $s \in S$ defines a real curve (that is, a knot) in M.

Proposition 5.6: Let $S \subset \mathrm{Kn}(M)$ be a surface of knots. All of its polar curves are homologous, and $[C_p] \in H^{1,1}(S)$ is the Kähler class.

Proof: Polar curves come by transgression along the evaluation map $\mathrm{ev}\colon S^\circ \to M$ from a point, and the Kähler class comes by a transgression of a fundamental class $[M] \in H^3(M, \mathbb{Z})$, which is Poincaré dual to the class of a point. ■

Proposition 5.7: Let S be a surface of knots. Then $h^2(S) \geqslant 2$.

Proof: In $H^2(S)$, one has the Kähler class and the Chern class $c_1(S^\circ \to S)$. The latter is nonzero by Proposition 5.3, and they wedge multiply to zero since $\int_S \omega \wedge c_1(S^\circ \to S) = [C_p].c_1(S^\circ \to S) = c_1(C_p^\circ \to C_p)$, and the latter class is zero since marking the point p defines a section of the circle bundle $C_p^\circ \to C_p$. ■

Remark 5.8: In particular, a surface of knots cannot be isomorphic to $\mathbb{C}P^2$.

Definition 5.9: Let $S \subset \mathrm{Kn}(M)$ be a compact complex surface. Its **index** is the degree of the velocity map $\mathrm{vel}\colon S^\circ \to \mathrm{Tw}(M)$.

In the final part of this paper, we classify the surfaces of index one. In order to do this, we need a following lemma.

Lemma 5.10: Let X be a compact projective surface polarized by a rational curve. Then X is either a projective plane, or a quadric, or a Hirzebruch surface.

Proof: ■

Theorem 5.11: Let M be a conformally Riemannian 3-dimensional manifold, and $S \subset \mathrm{Kn}(M)$ a compact complex surface of index one in its space of knots. Then S is a Hirzebruch surface $\mathbb{F}_{2k}$, and M is a quotient of a 3-shpere with its ????? metric.

Proof: All the polar curves are rational, which means that the surface S is swept by rational curves and is itself rational. Thus $h^{2,0}(S) = 0$, and $h^{1,1} = h^2 > 1$.

?????? ■

Acknowledgements. We are grateful to Lev Soukhanov, Misha Verbitsky, and Ivan Yakovlev for insightful discussions.

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