Isotrivial fibrations on generalised Kummer manifolds

Abstract: This note's aim is to prove that any isotrivial Lagrangian fibration on a deformation of a generalised Kummer manifold is a Tate–Shafarevich twist of the standard fibration $\mathrm{Kum}^n(E\times F)\to \mathrm{P}^n\cong\Sigma^{-1}(0)\subset S^{n+1}E$ on the generalised Kummer of a product of two elliptic curves. This refines the classification given by Kim, Laza, and Martin [KLM] by showing that Type B isotrivial fibrations never occur on deformations of generalised Kummer manifolds.

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1. Introduction

1.1. Let (Λ, q) be a lattice. The *divisibility* $d=\mathrm{div}(e)$ of $e\in\Lambda$ is the greatest common divisor of all the values of $q(e,h)$ for $h\in\Lambda$. It determines an element $e^*=q(e/d,-)\in\Lambda^*/\Lambda$. Its order r divides d, and $e=d/r\cdot e'$ for some primitive $e'\in\Lambda$. In particular, $\langle e^*\rangle\subset\Lambda^*/\Lambda$ is a cyclic group of order d iff $e\in\Lambda$ is primitive.

Definition 1.2. Let $\pi\colon X\to \mathrm{P}^n$ be a Lagrangian fibration on a holomorphically symplectic $2n$-fold. Its *isotropic class* is the class $\eta=c_1(\pi^*\mathcal{O}_{\mathrm{P}^n}(1))\in H^2(X,\mathbb{Z})$.

The isotropic class is indeed BBF-isotropic. Let us remind that the representation of $\mathrm{Kum}^n(A)$ as a fibre of the summation map $A^{[n+1]}\to A$ yields a Hodge isometry

$$H^2(\mathrm{Kum}^n(A),\mathbb{Z})\cong H^2(A,\mathbb{Z})\oplus\langle -2(n+1)\rangle,$$

where the new summand is spanned by δ, half the class of the exceptional divisor of the Hilbert–Chow map $A^{[n+1]}\to A^{(n+1)}$. From now on, Λ stands for the lattice $U^3\oplus\langle -2(n+1)\rangle\cong\big(H^2(\mathrm{Kum}^n,\mathbb{Z}),q_{BBF}\big)$.

1.3. An important question is which primitive isotropic classes are equivalent under the monodromy group $\mathrm{Mon}^2(X)\subset \mathrm{O}^+\big(H^2(X,\mathbb{Z}),q_{BBF}\big)$. The answer to it depends on the particular deformation type of X. For the case of the generalised Kummer manifolds, the two isotropic classes η,η' are related by a monodromy operator iff they have equal divisibility d and also

$$\eta^*=\pm\eta'^*\in\Lambda^*/\Lambda\cong\mathbb{Z}/(2n+2).$$

For generalised Kummer manifolds, divisibility d determines the polarisation type of the fibre as $(1,\dots,1,d,(n+1)/d)$. In particular, d^2 has to divide $n+1$ [Mon, W]. By Paragraph 1.1, the classes η^* of primitive ηs of divisibility d all lie in the cyclic subgroup $\langle(2n+2)/d\rangle \subset \mathbb{Z}/(2n+2)$, and are the generators of this subgroup.

Definition 1.4. Let (X,η) be as above. We call the Hodge structure on $\eta^\perp/\eta$ obtained from that on $H^2(X,\mathbb{Z})$ the *essential Hodge structure* of the fibration, and the quotient of the period domain $\mathrm{Per}(\eta^\perp/\eta)$ of all possible essential Hodge structures by the stabiliser $\mathrm{Mon}^2(X,\eta) \subset \mathrm{Mon}^2(X)$ on η in the monodromy group of X the *Markman–Markushevich space* $\mathrm{Mark}(X,\eta)$. The essential Hodge structures form a variation of weight two $\mathbb{Z}$-Hodge structures over the Markman–Markushevich space.

1.5. It is immediate that the essential Hodge structure does not depend on a Tate–Shafarevich deformation. The Markman–Markushevich space is a quasi-projective orbifold. These spaces of Lagrangian fibrations modulo Tate–Shafarevich twists were first defined by the author in [Dée] based on the classical works of the two namesake authors [Markm, Marku].

1.6. The usual construction of Debarre [Deb] takes an abelian surface (A,L) with polarisation of type $(1,n+1)$, for which generic member of $|L|$ is integral, and a number d. Its output is a holomorphically symplectic variety of deformation type Kum^n with a Lagrangian fibration over $|L|$, and fibre over $C \in |L|$ isomorphic to the connected component of $\ker\Big(\Sigma\colon \mathrm{Pic}^d(C) \to A\Big)$. The isotropic vector of the Debarre fibration is $c_1(L)-\delta$, it has divisibility one since $c_1(L)$ does.

1.7. To obtain fibrations with other polarisations, one has to use a similar construction of Yoshioka [Y]. For an abelian surface A, its *Mukai lattice* $H^{2\cdot}(A,\mathbb{Z})$ with pairing

$$(r,c_1,c_2)\cdot(r',c_1',c_2') = \int_A (c_1c_1' - rc_2' - r'c_2)$$

is unimodular, and any primitive Mukai vector $v \in H^{2\cdot}(A,\mathbb{Z})$ yields a moduli space $M(A,v)$ of sheaves with such Chern classes. The fibre of its Albanese map $M(A,v) \to A\times\hat{A}$ is a generalised Kummer-type manifold $K(A,v)$ of dimension v^2-2, and

$$H^2(K(A,v),\mathbb{Z}) \cong v^\perp \subset H^{2\cdot}(A,\mathbb{Z}).$$

For $v=(0,dc_1(L),b)$, where $L\to A$ is a primitive polarisation and $\gcd(d,b)=1$, the space $K(A,v)$ admits a Lagrangian fibration via the support map $K(A,v)\mapsto|dL|$. Under the identification $H^2(K(A,v),\mathbb{Z}) \cong v^\perp$, the isotropic vector of this fibration is $(0,0,1)$, and the essential Hodge structure is $c_1(L)^\perp \subset H^2(A,\mathbb{Z})$. This may be phrased in a way convenient for our purposes:

Proposition 1.8. Let $\eta\in\Lambda$ be a primitive isotropic vector of divisibility d. Then there exists an abelian surface (A,L) with polarisation of type $(1,(n+1)/d^2)$ such that the Albanese fibre $K(A,v)\subset M(A,v)$ of the Yoshioka moduli space of sheaves on A with Mukai vector $v=(0,dc_1(L),b)$ is smooth, and the support map $K(A,v)\to \mathrm{P}^n$ is a Lagrangian fibration with the isotropic vector from the orbit of η.

Proof: The smoothness is equivalent to $\gcd(d,b)=1$. The choices of $b \in (\mathbb{Z}/d)^\times$ correspond to choices of a generator of the subgroup $\mathbb{Z}/d \subset \Lambda^*/\Lambda$. □

We shall call these fibrations the *Debarre fibrations* as well.

2. Markman–Markushevich spaces for Kummer type

The importance of Markman–Markushevich spaces (Definition 1.4) stems from the theorem of, naturally, Markman (and Markushevich for $n=2$) which we put thus:

Theorem 2.1 ([Markm, Theorem 1.5]). The Beauville–Mukai construction which associates to a polarised K3 surface of genus n the compactified relative Pic^d of its hyperplane section yeilds an étale map $\mathcal{F}_n \to \mathrm{Mark}(\mathrm{K3}^{[n]}, \eta)$ for an appropriate vector $\eta \in H^2(\mathrm{K3}^{[n]}, \mathbb{Z})$. □

Certain divisors in the Markman–Markushevich spaces for $\mathrm{K3}^{[n]}$ type are omitted by the Beauville–Mukai map, and require taking a partial compactification of $\mathcal{F}_n$ for surjectivity. Recent important works [DMSh] and [HSh, Section 3] outline the parallel theory for the OG10 deformation type. An even stronger result for the generalised Kummer varieties follows almost immediately from the works of Yoshioka:

Theorem 2.2. Let $\eta \in H^2(\mathrm{Kum}^n, \mathbb{Z})$ be a primitive BBF-isotropic vector of divisibility d. Then the Debarre map $: \mathcal{A}_{1,(n+1)/d^2} \to \mathrm{Mark}(\mathrm{Kum}^n, \eta)$ is an orbifold isomorphism.

Proof: The Debarre variety $K(A,v)$ for $v=(0,dc_1(L),b)$, where L is of type $(1,(n+1)/d^2)$ on an abelian surface A, has essential Hodge structure which is Hodge isometric to that of $c_1(L)^\perp \subset H^2(A,\mathbb{Z})$. Hence the moduli space of abelian surfaces and the Markman–Markushevich space of the Lagrangian fibrations are uniformised by the same Siegel upper half-space (notice that for K3s this does not happen, since $\mathcal{F}_n$ is uniformised by a Cartan type IV domain minus a union of divisors orthogonal to (-2)-vectors, whereas on abelian surfaces (-2)-curves are not found). Therefore, the only thing to check are the fundamental groups.

Let $t=(n+1)/d^2$. The orbifold fundamental group of $\mathcal{A}_{1,t}$ is isomorphic to $\Gamma_t/\{\pm 1\}$ where Γ_t is the integral symplectic group which preserves the form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \\ -1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \end{pmatrix}.$$

The group $\mathrm{Mon}^2(\mathrm{Kum}^n,\eta) \subset \mathrm{Mon}^2(\mathrm{Kum}^2)$ is the subgroup of $\mathrm{O}^+(\Lambda)$ which acts through ± 1 on Λ^*/Λ with sign equal to the sign of the determinant of the element [Mon]. It does not act faithfully on $\mathrm{Per}(\eta^\perp/\eta)$ and do not give rise to a nice quotient of $\mathrm{Per}(\eta^\perp)$: its kernel is generated by the translations along the Tate–Shafarevich lines which have dense orbits. Let $M(\eta)$ be the image of $\mathrm{Mon}^2(\mathrm{Kum}^n,\eta)$ in $\mathrm{O}^+(\eta^\perp/\eta)$. Then the orbifold fundamental group of $\mathrm{Mark}(\mathrm{Kum}^n,\eta)$ is $M(\eta)$ or $M(\eta)/\{\pm 1\}$ depending on whether $-\mathrm{id}_{\eta^\perp/\eta} \in M(\eta)$ or not. In any way, $M(\eta)$ lies inside the subgroup of $\mathrm{O}^+(\eta^\perp/\eta)$ which acts through ± 1 on the discriminant of $\eta^\perp/\eta$.

The monodromy of the Debarre construction yields the map $\Gamma_t \to M(\eta)$ with kernel ± 1. To prove that it yields an isomorphism of fundamental groups, it is enough to

compare the indices of the two subgroups: $\Gamma_t/\{\pm 1\} \subset \mathrm{O}^+\big(c_1(L)^\perp\big)$ and $M(e)/(M(e) \cap \{\pm \mathrm{id}\}) \subset \mathrm{O}^+(\eta^\perp/\eta)$, as Yoshioka's and Mukai's machinery provide a natural Hodge isometry of sublattices $c_1(L)^\perp \subset H^2(A,\mathbb{Z})$ and $\eta^\perp/\eta \subset H^2(\mathrm{Kum}^n,\mathbb{Z})/\eta$. In the first case, the group $\Gamma_t/\{\pm 1\}$ is isomorphic to $\widehat{\mathrm{SO}^+}\big(c_1(L)^\perp\big)$, that is, the subgroup of $\mathrm{SO}^+\big(c_1(L)^\perp\big)$ which acts trivially on the discriminant group of $c_1(L)^\perp \subset H^2(A,\mathbb{Z})$. For squarefree t, this is [GH, Proposition 1.2], generalised to any t along the lines of [GHS, Theorem 5.7]. The monodromy of the Debarre construction thus yields the inclusions

$$\widehat{\mathrm{SO}^+}\big(\eta^\perp/\eta\big) \subseteq M(\eta) \subseteq \widehat{\mathrm{SO}^+}\big(\eta^\perp/\eta\big) \cdot \big\{\pm \mathrm{id}_{\eta^\perp/\eta}\big\} \subset \mathrm{O}^+\big(\eta^\perp/\eta\big).$$

In particular, $M(\eta)$ contains an element with determinant -1 iff it contains $-\mathrm{id}_{\eta^\perp/\eta}$, which means that $M(\eta)/\left(M(\eta) \cap \{\pm \mathrm{id}_{\eta^\perp/\eta}\}\right) \cong \widehat{\mathrm{SO}^+}\big(\eta^\perp/\eta\big) \cong \Gamma_t/\pm 1$ in any case. □

Corollary 2.3. Let $X \to \mathrm{P}^n$ be a Lagrangian fibration on a Kum^n-type manifold. There exists a polarised abelian surface (A,L) and $d,b \in \mathbb{Z}$ such that $X \to \mathrm{P}^n$ is birational to a Tate–Shafarevich twist of $K(A,v) \to |dL|$ for $v = (0, dc_1(L), b) \in H^{2\cdot}(A,\mathbb{Z})$.

Proof: Let $\eta \in H^2(X,\mathbb{Z})$ be the isotropic vector of the fibration. By Theorem 2.2, there exists a polarised abelian surface (A,L) which has $c_1(L)^\perp \cong \eta^\perp/\eta$. Then η can be identified with the vector $(0,0,1) \in H^{2\cdot}(A,\mathbb{Z})$, perhaps after replacing A with $\hat{A}$. By the assumptions, X yields a point in the period domain of Kum^n-type manifolds which belongs to the same Tate–Shafarevich line as $K(A,v)$. Tate–Shafarevich deformations of Kähler holomorphically symplectic manifolds are Kähler [A], hence a deformation $K(A,v)^\varphi$ corresponding to the same point in the period space as X is birational to X thanks to a theorem of Huybrechts. □

3. Application to isotrivial fibrations

Theorem 3.1. An isotrivial Lagrangian fibration on a deformation of a generalised Kummer manifold is a Tate–Shafarevich twist of a standard Kummer fibration.

Proof: By Corollary 2.3, such a fibration is birational to a Tate–Shafarevich twist of a Debarre fibration on a certain polarised abelian surface (A,L). The fibres of a Debarre fibration are the kernels of the maps $\mathrm{Jac}(C) \to A$ for the curves $C \in |dL|$. If C has an irreducible component of genus greater than two, it deforms within the linear system, which gives a nontrivially varying factor of the fibre. Hence all the components of curves in $|dL|$ have genus 1 or 2. If D is a genus 2 component and E a genus 1 component, it's easy to see from Riemann–Roch that linear systems $|2D|$ and $|D+E|$ have smooth members of genera 5 and 4, respectively, hence the only possibility for a genus 2 component is $d=1$ and $L = \mathcal{O}(D)$, the case of a Jacobian Kummer surface.

Therefore, A is a product of elliptic curves $E \times F$. The linear system $|2E+2F|$ has smooth genus 5 members, hence $dL = \mathcal{O}(E+(n+1)F)$ and $d=1$. The Debarre fibration in this case coincides with the standard Kummer fibration $\mathrm{Kum}^n(E\times F) \to S_0^{n+1}E$. □

Corollary 3.2. A Lagrangian fibrations on deformation of a generalised Kummer manifold with divisible isotropic vector is never isotrivial.

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