Period map for coassociative fibrations

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Abstract
$\boxed{\text{Boxes}}$ denote the questions to be investigated.

Contents

1 Introduction

The purpose of this note is to investigate possible generalisations of period maps to coassociative fibrations on G_2-manifolds.

2 Reminder

2.1 Reminder: period maps for elliptic fibrations

Let (S,σ) be a K3 surface with its holomorphic symplectic form, and $S \xrightarrow{\pi} B \cong \mathbb{C}\mathrm{P}^1$ is a Lagrangian fibration. The standard algebraic-geometric machinery yields a variation $F_{\mathbb{Z}} = F^1_{\mathbb{Z}} = R^1\pi_*\mathbb{Z}_S$ of Hodge structures on $B^\circ = B \setminus \Delta$, $\Delta \subset B$ being the set of degenerate fibres, which represents the cohomology of the fibres. Let us outline its differential-geometric interpretation. For any vector $v \in T_bB$ one may define a 1-form $\alpha_v = (\iota_{\tilde{v}}\sigma)|_{S_b}$, where $\tilde{v} \in \Gamma(TS|_{S_b})$ is a lift of v (the result is independent of its choice thanks to $S_b \subset S$ being holomorphically Lagrangian). This is a closed $(1,0)$-form on the genus one curve S_b. This yields the isomorphism of vector bundles $T(B^\circ) \to F^{1,0} \subset F_{\mathbb{C}}$. The bundle $F_{\mathbb{C}}$, however, is a complexification of the local system $F_{\mathbb{Z}}$, which means that it has a flat connection with integral monodromy (called the *Gauss–Manin* connection). Choose a universal cover $\tilde{B}$ of B°, then the Gauss–Manin connection defines a trivialisation of $\tilde{F_{\mathbb{C}}}$, and $\tilde{B}$ gets a map to the *period domain* $\mathrm{Per} = \mathrm{P}\{\alpha \in H^1(S_b,\mathbb{C})\colon -\sqrt{-1}\int_{S_b}\alpha\wedge\overline{\alpha} > 0\}$ (which, in particular, endows B° with a non-complete *Weil–Petersson* Hermitian metric $|v|^2_{WP} = \frac{1}{2\sqrt{-1}}\int_{S_b}\alpha_v\wedge\overline{\alpha_v}$).

Proposition 2.1: The period map is holomorphic.

Proof: $\boxed{\ldots}$ ■

At the same time, post-composing this isomorphism with the projection $F^{1,0} \to F^1_{\mathbb{R}}$ yields an isomorphism of $T(B^\circ)$ with the flat cohomology bundle, and thus a non-holomorphic flat connection on B° (depending on the context, it may be referred to as the Gauss–Manin or *Liouville–Arnold* connection).

2.2 G_2-manifolds and coassociative fibrations

Definition 2.2: A *seven-dimensional vector product* is defined by $u \times v = \mathrm{Im}(uv)$, where $u, v \in \mathrm{Im}\,\mathbb{O}$ are purely imaginary octonions. The automorphism group of this tensor, called $G_2 \subset \mathrm{GL}(7, \mathbb{R})$ preserves one and only one nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, which allows to pack the vector product into a 3-form $\rho(u, v, w) = \langle u \times v, w \rangle$. This is one of the special holonomy groups: a Riemannian manifold is said to be a *G_2-manifold* iff its Riemannian holonomy is contained in a copy of G_2, or, equivalently, it carries a *fundamental 3-form* ρ which is closed and at each point can be identified with the standard 3-form defined above via a linear isometry.

Example 2.3: Let $\mathbb{H}$ be a quaternionic line acted by complex structure operators I, J, K with standard relations. Then $\langle I, J, K \rangle \oplus \mathbb{H}$ has a G_2-structure defined uniquely by the properties $S \times u = Su$ for any $S \in \langle I, J, K \rangle$ and $u \in \mathbb{H}$, $S \times S' = -SS'$ for $S, S' \in \langle I, J, K \rangle$, and $x \times (x \times y) = -|x|^2 y$ for any x, y. The fundamental 3-form on this space is

$$\rho = \varepsilon_I \wedge \omega_I + \varepsilon_J \wedge \omega_J + \varepsilon_K \wedge \omega_K - \varepsilon_I \wedge \varepsilon_J \wedge \varepsilon_K$$

where ε_S stands for the 1-form dual to a unit vector $S \in \langle I, J, K \rangle$, and ω_S for the Kähler form on $\mathbb{H}$ for the complex structure S.

Definition 2.4: A subspace U in a G_2-space (that is, a real vector space with seven-dimensional cross product) is called *associative* iff it has dimension three and is closed under cross-product, and *coassociative* iff it has dimension four, and for any $u, v \in U$ one has $u \times v \perp U$. Associative and coassociative subspaces are orthogonals of each other. A submanifold in a G_2-manifold is called (co)associative iff all of its tangent spaces are.

Example 2.5: Any pair of orthogonal associative and coassociative subspaces $V \oplus^\perp U$ yields a quaternionic structure on U via $S(u) = S \times u$, $S \in V$, $u \in U$. In particular, any coassociative submanifold carries an almost hypercomplex structure. Notice that the cross multiplication of vectors in V is the opposite of the multiplication of the corresponding imaginary quaternions. A subspace $\langle S, u, Su \rangle, S \in V, u \in U$ is therefore associative; similarly, span of any S-complex line in U and $S^\perp \subset V$ is coassociative.

Example 2.6: Let Y be a K3 surface with its three complex structures I, J, K. Take a torus $T^3 = \langle I, J, K \rangle / \Lambda$. If $\omega_I, \omega_J, \omega_K$ are the Kähler forms on Y, and η_I, η_J, η_K are three co-ordinate one-forms on T^3, the 3-form

$$\varphi = \omega_I \wedge \eta_I + \omega_J \wedge \eta_J + \omega_K \wedge \eta_K - \eta_I \wedge \eta_J \wedge \eta_K$$

is a closed G_2-form on the product $Y \times T^3$. The corresponding Riemannian metric is the Ricci-flat metric on Y times the (Ricci-)flat metric on T^3. If $\Sigma \subset Y$ is an I-complex curve, and Λ is a lattice such that $\Lambda \cap \langle J, K\rangle$ is also a lattice of full rank (that is, there exists a map of tori $T^2 \to T^3 \to S^1 \cong \langle I\rangle/\mathbb{Z}$), then $\Sigma \times T^2 \subset Y \times T^3$ is a coassociative submanifold. If (Y, I) admits a holomorphic fibration $Y \to S^2 \cong \mathbb{C}\mathrm{P}^1$, then the product map $Y \times T^3 \to S^2 \times S^1$ is a coassociative fibration.

If $F \subset X$ is a coassociative submanifold, the map $v \mapsto (\iota_v \rho)\,|_F$ is an isomorphism of the vector bundles $\nu_{F/X} \xrightarrow{\sim} \Lambda^2_- T^*F$ ([McLean, Proposition 4.2]). Moreover, McLean proves that a Jacobi vector field for a one-parametric family of coassociative submanifolds corresponds to closed 2-forms on F. Therefore, a fibre of a coassociative fibration is an almost hypercomplex manifold endowed with three symplectic forms. However, this is not a hyperkähler structure except for the case of a product (in which the holonomy group is a proper subgroup of G_2): the unit normal vector fields (which yield almost complex structures via cross product) and Jacobi vector fields are not the same thing.

Let us try to emulate the construction of a period map for G_2-manifolds. If $X \xrightarrow{\pi} M$ is a coassociative fibration on a G_2-manifold X with fundamental 3-form ρ, one obtains a map $T_b B \to H^2(X_b, \mathbb{R})$ via $v \mapsto [\iota_{\tilde{v}}\rho]$. This is an embedding of real vector bundles $T(B \setminus \Delta) \to E^2_{\mathbb{R}} = R^2\pi_*\underline{\mathbb{R}}$ into the second cohomology bundle. The Poincaré pairing endows E with an inner product, which is negative definite on the image of T (that is, the negative eigensubbundle for the Hodge star operator). The Gauss–Manin connection on $E_{\mathbb{R}}$ yields a connection ∇ in T which is metric but not flat anymore.

Proposition 2.7: The connection ∇ is the Levi-Civita connection of the Weil–Petersson metric.

Proof: Let us compute its torsion. If ψ is a differential form on the total space whose restrictions $\psi|_{X_b}$ to each fibre is closed, then the Gauss–Manin derivative $\nabla^{GM}_v([\psi|_{X_b}])$ may be expressed as the section $b \mapsto [(L_{\tilde{v}}\psi)|_{X_b}]$ for any lift of the basic vector field v. In our case, one may write $\tau(u, v) = \nabla_u v - \nabla_v u - [u, v] = [(L_{\tilde{u}}\iota_{\tilde{v}}\rho)|_{X_b}]^- - [(L_{\tilde{v}}\iota_{\tilde{u}}\rho)|_{X_b}]^- - \left[\left(\iota_{\widetilde{[u,v]}}\rho\right)|_{X_b}\right]$, where a^- stands for the anti-self-dual part of the class $a \in H^2(X_b, \mathbb{R})$. The difference $\widetilde{[u, v]} - [\tilde{u}, \tilde{v}]$ is vertical and hence, when substituted into ρ and restricted to the fibre, vanishes identically, so the last term may be replaced with $\left[\left(\iota_{[\tilde{u},\tilde{v}]}\rho\right)|_{X_b}\right] = [(L_{\tilde{u}}\iota_{\tilde{v}}\rho - \iota_{\tilde{v}}L_{\tilde{u}}\rho)|_{X_b}] = [(L_{\tilde{u}}\iota_{\tilde{v}}\rho - \iota_{\tilde{v}}d\iota_{\tilde{u}}\rho)|_{X_b}]$. Subtraction yields $\tau(u, v) = -\left[(L_{\tilde{u}}\iota_{\tilde{v}}\rho)|_{X_b}\right]^+ - [(d\iota_{\tilde{v}}\iota_{\tilde{u}}\rho + \iota_{\tilde{v}}d\iota_{\tilde{u}}\rho)|_{X_b}]^- + [(\iota_{\tilde{v}}d\iota_{\tilde{u}}\rho)|_{X_b}] = [(d\iota_{\tilde{u}}\iota_{\tilde{v}}\rho)|_{X_b}]^- - [(\iota_{[\tilde{u},\tilde{v}]}\rho)|_{X_b}]^+ = 0 - 0 = 0.$ ■

Let us propose the following candidate for the period map.

Definition 2.8: Assume all fibres of the coassociative fibration $X \to B$ are smooth and its base simply connected (otherwise simply replace B by the universal cover of $B \setminus \Delta$ or merely a small ball in B). The *period map* is defined on the space ST^*B, the spherisation of the cotangent bundle. If $(b, \tau \subset T_b B)$ is an oriented tangent plane and $\tau = \langle u, v\rangle$ is a positively oriented orthonormal basis, define $\mathfrak{per}(b, \tau) = \langle[\omega_u + \sqrt{-1}\omega_v]\rangle \subset \mathrm{P}(H^2(X_b, \mathbb{C}))$.

The target space is, at least, a complex manifold—namely, the period domain, well-known in the hyperkähler geometry.

2.3 (Almost) CR twistor spaces

For a 3-manifold M, the space ST^*M carries a natural contact distribution H. Any surface $N \subset M$ lifts to a Legendrian surface of this distribution. Therefore, a choice of a complex structure operator on H yields a condition on points on surfaces—namely, the Legendrian lift being holomorphic at this point. Here are two examples:

Example 2.9: Let M be a Riemannian 3-manifold. Let us split H with the Levi-Civita connection, and let $V_{\pi,x} \cong \mathrm{Hom}(\pi, T_x/\pi)$ and $H/V \cong \pi$ be endowed with the standard complex structure operator. The resulting almost CR structure on (ST^*M, H) is CR-integrable and only depends on the conformal class of the metric. The Gaussian lift of a surface is holomorphic at a point iff this point is umbilic in the surface (in particular, it has very few pseudoholomorphic Legendrian surfaces). This is called the **LeBrun's CR twistor space** of a conformal 3-manifold.

Example 2.10: In the same situation as above, switch the orientation of the complex structure on the vertical bundle. This new almost CR structure is neither CR-integrable nor conformally invariant, but it has a lot of pseudoholomorphic Legendrian surfaces, as the Gaussian lift in this structure is holomoprhic precisely at the point where the mean curvature vector vanishes. This is called **Eells–Salamon almost CR structure** (or, rather, we shall call it that after the same structure on twistors of Riemannian 4-manifolds).

This begs a
Question 2.11: For which almost CR structure on ST^*B of the base B of the coassociative fibration the period map $ST^*B \to \mathrm{Per}(X_b)$ is CR holomorphic?

3 Palpable examples

3.1 Surface times a torus

Take $(Y, I) \to S^2$ a holomorphically symplectic surface with an elliptic fibration (say, an elliptic K3 or a rational elliptic surface with one fibre removed), and $T^3 = \mathbb{R}\langle I, J, K\rangle/\Lambda$ a torus such that $\mathrm{rk}_{\mathbb{Z}}(\mathbb{R}\langle J, K\rangle \cap \Lambda) = 4$. As we have noticed in the Example 2.6, the product map $Y \times T^3 \to S^2 \times S^1$ is a coassociative fibration with fibres $Y_b \times T^2$. Let us compute its Weil–Petersson metric and period map.

Let $u \in T_{b\times t}(S^2 \times S^1)$ be a vector tangent to the S^1-multiple. The corresponding 2-form ω_u on $X_{b\times t} = Y_b \times T^2$ is just the sum of the pullbacks of the Kähler form on Y_b and the area form on T^2. Since the degree of Y_b is independent of b, the S^1-component of $S^2 \times S^1$ is of constant length. Moreover, the parallel translations in the S^1-direction is a Weil–Petersson isometry.

Now let $v \in T_{b\times t}(S^2 \times S^1)$ be tangent to S^2-component. The formula for ρ yields

$$\iota_{\tilde{v}}\rho = \iota_{\tilde{v}}\omega_I \wedge \eta_I + \iota_{\tilde{v}}\omega_J \wedge \eta_J + \iota_{\tilde{v}}\omega_K \wedge \eta_K.$$

When restricted to the fibre, the first term vanishes, so one has

$$\omega_v = (\mathrm{Re}\ \alpha_v) \wedge \eta_J + (\mathrm{Im}\ \alpha_v) \wedge \eta_K,$$

and $g(v,v) = \frac{1}{2\sqrt{-1}} \int_{Y_b \times T^2} \alpha_v \wedge \overline{\alpha_v} \wedge \eta_J \wedge \eta_K = |v|^2_{WP} \cdot \mathrm{Area}(T^2)$. It is also easy to see that for u and v as above the product $\omega_u \wedge \omega_v$ vanishes identically. In other words, $(S^2 \times S^1, g)$ is isometric to the orthogonal product of S^2 with Weil–Petersson metric induced by the fibration $Y \to S^2$ scaled by the area of the torus, and a circle of fixed length.

Künneth's formula yields a Gauss–Manin-parallel decomposition

$$E^2_{\mathbb{R}} = F^0_{\mathbb{R}} \otimes H^2(T^2, \mathbb{R}) \oplus F^1_{\mathbb{R}} \otimes H^1(T^2, \mathbb{R}) \oplus F^2_{\mathbb{R}} \otimes H^0(T^2, \mathbb{R}).$$

The negative subbundle, that is, the image of $T(S^2 \times S^1)$, splits into the Gauss–Manin-parallel line subbundle spanned by $\omega_I - \eta_J \wedge \eta_K$ (that is, the tangent bundle to circles), which lies inside the direct sum of the zeroth and second subbundles, and everything else, which is a rank two subbundle of $F^1_{\mathbb{R}} \otimes H^1(T^2, \mathbb{R})$.

The rotations in the S^1-directions act by contactomorphisms of ST^*B, and the quotient of ST^*B by this action is $S^2 \times S^2$. The period map $ST^*B \to \mathrm{Per}(T^4)$ factorises through this fibration. On the vertical S^2s, it is the embedding of twistor lines into the period domain. On the horizontal S^2s, let us take a vector v and $Iv = w$ (so that $\mathrm{Re}\ \alpha_w = -\mathrm{Im}\ \alpha_v$ and $\mathrm{Im}\ \alpha_w = \mathrm{Re}\ \alpha_v$). In this case $[\omega_v + \sqrt{-1}\omega_w] = [\mathrm{Re}\ \alpha_v \wedge \eta_J + \mathrm{Im}\ \alpha_v \wedge \eta_K] + \sqrt{-1}\,[\mathrm{Re}\ \alpha_w \wedge \eta_J + \mathrm{Im}\ \alpha_w \wedge \eta_K] = \mathrm{Re}\ \alpha_v \wedge \big(\eta_J + \sqrt{-1}\eta_K\big) + \mathrm{Im}\ \alpha_v \wedge \big(\eta_K - \sqrt{-1}\eta_J\big) = \overline{\alpha_v} \wedge \big(\eta_J + \sqrt{-1}\eta_K\big)$. Thus the period map restricted to a horizontal S^2 is the *complex conjugate* of the period map for the elliptic fibration $Y \to S^2$ (post-composed with the inclusion of the upper half-plane to the Siegel upper-half space via the multiplication by the fixed elliptic curve $\big(T^2, \eta_J + \sqrt{-1}\eta_K\big)$). In other words, one have the following

Proposition 3.1: The period map for a coassociative fibration $Y \times T^3 \to S^2 \times S^1$ from Example 2.6 is anti-holomorphic with respect to the Eells–Salamon almost CR structure on $ST^*(S^2 \times S^1)$. ■

Instead of proving this statement for an arbitrary coassociative fibrations, we make it into a definition.

Definition 3.2: A coassociative fibration $(X, \rho) \to B$ is called *civilised* iff the period map $ST^*B \to \mathrm{Per}(X_b)$ is anti-holomorphic for the Eells–Salamon almost CR structure associated with the Weil–Petersson metric on B°.

3.2 Twistor deformations of products of elliptic curves

Let us remind first that the period domain has two interpretations: as a domain in a complex quadric $\mathrm{P}\{v \in V \otimes \mathbb{C} \colon q(v,v) = 0, -\sqrt{-1}q(v,\overline{v}) > 0\}$, or as the subset $\mathrm{Gr}_{++}(V_{\mathbb{R}}, q) = \{\pi \in \mathrm{Gr}(2, V) \colon q|_\pi > 0\}$ in the oriented Grassmannian. As a homogeneous space, the latter is $\mathrm{SO}(n,m)_0 / [\mathrm{SO}(2) \times \mathrm{SO}(n-2, m)]$, the isomorphism being $v \mapsto \mathrm{span}\langle v, \overline{v}\rangle$.

Let $A \in \mathrm{Per}$ be an abelian surface. A twistor line through A is not unique: it depends on the choice of a Kähler class on A. In particular, each point in Per has a real three-dimensional family of twistor curves passing through it. Therefore in general it is not reasonable to speak of the variety swept by twistorial curves passing through a given family of abelian surfaces. However, in our case there is a natural choice of a Kähler class: it is the form ω_I which is the orthogonal product of unit-area form on Y_b and unit-area form on T^2.

Proposition 3.3: Let $T^4 = T^2 \times T^2$ be a product of two 2-tori, and $\omega = \pi_1^*\omega_{T^2} + \pi_2^*\omega_{T^2}$ be the polarisation. Then the union of all twistor lines for ω which pass through the points of the form $E_1 \times E_2 \in \mathrm{Per}$, where E_i are different complex structures on T^2 (so-called "Humbert surface" in the period domain) is a union of an open subset in a hyperplane section of Per and a real three-dimensional locus.
Proof: The holomorphic symplectic form on the surface that corresponds to a point in the said union is a linear combination of ω, $\sigma = \pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2$, and $\overline{\sigma}$, where α_i are the abelian differentials on E_i. Notice that the form $\tilde{\omega} = \pi_1^*\omega_{T^2} - \pi_2^*\omega_{T^2}$ wedge multiplies to zero with both. Hence the union lies inside the intersection $\mathrm{Per} \cap \tilde{\omega}^\perp \subset \mathbb{C}\mathrm{P}^5$. This is also a homogeneous domain, now of the form $\mathrm{SO}(3,2)_0/\left[\mathrm{SO}(2) \times \mathrm{SO}(1,2)\right]$ since $\tilde{\omega} \wedge \tilde{\omega} < 0$.

Notice that the intersection $\omega^\perp \cap \tilde{\omega}^\perp \cap \mathrm{Per}$ contains two components of which one is the Humbert surface itself: indeed, for a class $\sigma \in H^2(T^4, \mathbb{C}) \cong H^2(T^2) \oplus \left[H^1(T^2) \otimes H^1(T^2)\right] \oplus H^2(T^2)$ the first two conditions say that the first and the third components of this decomposition vanish, $\sigma \wedge \sigma = 0$ yields that $\sigma = \pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2$ for some 1-forms on T^2, and $\sigma \wedge \overline{\sigma} > 0$ tells that α_i are abelian differentials for some complex structures on T^2 which have the same orientation (if both orientations are wrong, we're in the other component). Notice that the subspace $\omega_{\mathbb{R}}^\perp \cap \tilde{\omega}_{\mathbb{R}}^\perp$ is of the signature $(2,2)$ with respect to the intersection form, so its Gr_{++} is isomorphic to

$$\mathrm{SO}_0(2,2)/\,\mathrm{SO}(2) \times \mathrm{SO}(2) \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2) \times \mathrm{SO}(2) \cong H^2 \times H^2$$

as expected. The twistor line for $E_1 \times E_2$ inside Per is cut out by the 3-subspace $\langle \sigma, \omega, \overline{\sigma} \rangle$ where $\sigma = \pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2$. Two such subspaces $\langle \sigma, \omega, \overline{\sigma} \rangle$ and $\langle \sigma', \omega, \overline{\sigma}' \rangle$ have either one-dimensional intersection spanned by ω (generic case) or two-dimensional intesection $\langle \omega, \kappa \rangle$ where $\kappa \in \omega^\perp \cap \tilde{\omega}^\perp$ is a real vector with $\kappa \wedge \kappa > 0$. Thus the union of the twistor lines contain the locus of the vectors $\langle \omega + \sqrt{-1}\kappa \rangle$ where different twistor lines from this family meet. This locus is of real dimension three, since choice of κ is a choice of a positive direction in the real 4-space $\omega^\perp \cap \tilde{\omega}^\perp \subset H^2(T^4, \mathbb{R})$ of signature $(2,2)$. Away from this locus, there is a unique twistor line through a point. It is easier to visualise in the real picture: a plane $\pi \in \mathrm{Gr}_{++}(\tilde{\omega}_{\mathbb{R}}^\perp)$ is a $H^{2,0} \oplus H^{0,2}$ for some twistor deformation of a product surface iff either $\omega \in \pi$ (this is the case we've already considered), or $\omega \notin \pi$ and the 3-subspace $\mathrm{span}\langle \omega, \pi \rangle$ is positive definite. In this case, the projection along the twistor lines is defined by $\pi \mapsto \omega^\perp \cap \mathrm{span}\langle \omega, \pi \rangle \in \mathrm{Gr}_{++}(\omega_{\mathbb{R}}^\perp \cap \tilde{\omega}_{\mathbb{R}}^\perp)$.

This open locus has the following interpretation: for a positive-definite subspace $\pi \not\ni \omega$ its $\mathrm{span}\langle \omega, \pi \rangle$ is also positive-definite iff the form $B(u,v) = u \wedge v - (u \wedge \omega)(v \wedge \omega)$ is also positive-definite on π. Therefore, this locus may be represented as $\mathrm{Gr}_{++}(\tilde{\omega}_{\mathbb{R}}^\perp, B)$. The form B is degenerate and ω spans its kernel, hence the quotient map by ω induces the fibration $\mathrm{Gr}_{++}(\tilde{\omega}_{\mathbb{R}}^\perp, B) \to \mathrm{Gr}_{++}(\omega_{\mathbb{R}}^\perp \cap \tilde{\omega}_{\mathbb{R}}^\perp)$. Its fibre is an open subset of the twistor line, a pair of discs separated by a circle, and this circle maps to the real three-dimensional boundary of the union of twistor lines (this relative circle along the Humbert surface is a real five-dimensional CR boundary of the open stratum, but when mapped to the period domain itself, it lands onto a real three-dimensional boundary by collapsing a family of complex curves). ■

3.3 Digression on adiabatic limits

For a coassociative fibration $X \xrightarrow{\pi} B$, take a surface $\Sigma \subset B$. The preimage $\pi^{-1}\Sigma \subset X$ is a hypersurface in a G_2-manifold, and thus a nearly Kähler manifold. The fibres are pseudoholomoprhic in it. On the other hand, if the Gaussian lift $\tilde{\Sigma} \subset ST^*B$ is pseudoholomorphic and thus maps to a complex curve in $\mathrm{Per}(X_b)$, this yields a different family of complex structures on the fibres over Σ. Of course, the two complex structures are not related since a nontrivial coassociative fibration is never a Riemannian submersion, but what if we close our eyes and pretend they are?

This would mean that the condition "$\pi^{-1}\Sigma \subset X$ is a complex manifold" and "$\tilde{\Sigma} \subset ST^*B$ is pseudo-holomorphic" are equivalent. The first, thanks to a theorem of Calabi, is a statement about the second fundamental form of $\pi^{-1}\Sigma$: it should be complex antilinear, in particular, its mean curvature vanishes. Along the fibre this is automatic since it is already coassociative; thus the first condition is equivalent to "$\Sigma \subset B$ is minimal." The Gaussian lifts of minimal surfaces are pseudoholomorphic precisely for the Eells–Salamon almost CR structure. Hence, all the coassociative fibrations are civilised.

Unfortunately, this proof relies on the claim "a coassociative fibration is a Riemannian submersion," which is not true. However, it is asymptotically true in the adiabatic limit. ??????????????????

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3.4 Fano threefolds and Kovalev's examples

3.5 Gukov–Yau–Zaslow's examples?

The Eells–Salamon almost CR structure is never CR integrable. This means that for a civilised coassociative fibration, the derivative of the period map always has kernel of rank at least one (as it is an anti-holomorphic map to an honest complex manifold). When this kernel is generically of rank one, we call the corresponding foliation into curves the *characteristic foliation.*

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References

[McLean]